MATHEMATICS

for Elementary School Teachers 6e

BASSAREAR • MOSS

Four Steps for Solving Problems

Understanding the Problem

Questions that can be useful to ask:

1. Do you understand what the problem is asking for?
2. Can you state the problem in your own words, that is, paraphrase the problem?
3. Have you used all the given information?
4. Can you solve a part of the problem?

Actions that can be helpful:

1. Reread the problem carefully. (Often it helps to reread a problem a few times.)
2. Try to use the given information to deduce more information.
3. Plug in some numbers to make the problem more concrete, more real.

Devising a Plan

Several common strategies:

1. Represent the problem with a diagram (carefully drawn and labeled).
   Check to see if you used (the relevant) given information. Does the diagram "fit" the problem?
2. Guess–check–revise (vs. "grope and hope"). Keep track of "guesses" with a table.
3. Make an estimate. The estimate often serves as a useful "check." A solution plan often comes from the estimation process.
4. Make a table (sometimes the key comes from adding a new column).
5. Look for patterns—in the problem or in your guesses.
6. Be systematic.
7. Look to see if the problem is similar to one already solved.
8. If the problem has "ugly" numbers, you may "see" the problem better by substituting "nice" numbers and then thinking about the problem.
9. Break the problem down into a sequence of simpler "bite-size" problems.
10. Act it out.

Carrying Out the Plan

1. Are you keeping the problem meaningful or are you just "groping and hoping?" On each step ask what the numbers mean. Label your work.
2. Are you bogged down? Do you need to try another strategy?

Looking Back

1. Does your answer make sense? Is the answer reasonable? Is the answer close to your estimate, if you made one?
2. Does your answer work when you check it with the given information? (Note that checking the procedure checks the computation but not the solution.)
3. Can you use a different method to solve the problem?

This sheet is meant to serve as a starting point. The number of strategies that help the problem-solving process are almost endless and vary according to each person's strengths and preferences.

After you solve a problem that was challenging for you or after you find that your answer was wrong, stop and reflect. Can you describe what you did that got you unstuck or things you did that helped you to solve the problem? If your answer was wrong, can you see what you might have done? *It is the depth of these reflections that connects to your increased ability to solve problems.*

Mathematics for Elementary School Teachers

6e

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***Mathematics for Elementary School Teachers,* Sixth Edition**

Tom Bassarear, Meg Moss

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ABOUT THE AUTHORS

Tom Bassarear

I have been teaching the Mathematics for Elementary School Teachers course for more than 20 years, and in that time, I have learned as much from my students as they have learned from me. This text was inspired by my students and reflects one of the most important things we have taught one another: that building an understanding of mathematics is an active, exploratory process, and ultimately a rewarding, pleasurable one. My own experience with elementary schoolchildren and my two children, Emily and Josh, has convinced me that young children naturally seek to make sense of the world they live in and for a variety of reasons many people slowly lose that curiosity over time. My hope is that this book will engage your curiosity about mathematics once again.

Meg Moss

I am excited and honored to be working with Tom Bassarear on this book. I began teaching Mathematics for Elementary Teachers over 20 years ago. I immediately began seeking advice from others who had taught the course, and volunteered in elementary classrooms to learn more. Teaching these courses has deepened my mathematical understanding as well as my understanding of how people learn math. Helping future elementary school teachers to truly understand mathematics, and see the beauty in mathematics is very rewarding, and I know that all of this will have a major positive impact on their future students. I appreciate you sharing this journey with me!

NEW TO THE SIXTH EDITION!

I am pleased to welcome Meg Moss, Ph.D., to this textbook and to introduce her to you. I have known Meg for about 10 years through conversations and presentations at conferences. I have admired the quality of her work, depth of thought, and commitment to students, so I was delighted when she agreed to join me in continuing this book as I transition toward retirement. While I have been actively involved in the current revision, you will find Meg's footprints throughout the book.

She has done a magnificent job of framing the Common Core State Standards in mathematics (CCSM), which have been adopted by 45 states, in a way that helps readers to see where their future students will learn these concepts and to help them see the importance of such concepts. The CCSM articulates eight mathematical practices (MPs) that replace standards by the National Council of Teachers of Mathematics (NCTM). While I believe that the NCTM standards are more clearly stated and more user-friendly, the eight MPs correlate strongly with NCTM's framework. Meg's genius was not to try to incorporate all the details of the MPs, which would be overwhelming, but to focus on and articulate the big ideas embedded in those practices. She articulates them in the first chapter, connecting them directly to Investigations, and then refers to them in appropriate ways throughout the book.

After many discussions, she constructed a revised and streamlined Chapter 1, which I love. She integrated the number theory concepts—which previously comprised a separate chapter—into the textbook, as those concepts are needed. This connects to research on learning that indicates students are more willing and able to retain ideas if they see how they are connected and if they use them immediately.

With CCSM's emphasis on algebraic thinking, we decided to have a separate Chapter 6 on algebra. Meg did a heroic job of researching best practices in algebra in schools and then organized the material into a coherent framework that addresses the important algebraic ideas articulated by CCSM. She took many Investigations from the fifth edition's algebra section and some from Chapter 1 and has added many of her own.

Meg also wrote Questions to Summarize Big Ideas for the end-of-chapter summaries. These questions help students reflect on what they have learned and articulate major "take-away" ideas from the chapter, ultimately supporting one of the most important ideas of the textbook—this is to OWN knowledge.

In addition, Meg went through every page of the textbook and you will see her work in many places, such as in

- revising text to make points more clear and concise;
- adding extra steps and more concreteness when she felt it would be helpful, especially for students who tend to struggle with those ideas;
- more visual representations including Singaporean bar models; and
- more technology, including references to virtual manipulatives, Geogebra investigations, and several other websites.

I hope you will welcome and appreciate Meg's contributions to the sixth edition as much as I do.

—Tom Bassarear

ANNOTATED CONTENTS

Chapter 1 Foundations for Learning Mathematics

This chapter continues the theme from the fifth edition, but with a new emphasis on the Mathematical Practices of the Common Core State Standards (CCSS). While references to the NCTM standards remain, one of the goals of the revised Chapter 1 is to lay the groundwork for the CCSS so that students can see some of those standards "in action" while they are learning the mathematics throughout the textbook. The explorations in the *Explorations* manual offer diverse types of problems to grapple with that support the strategies used in the rest of the course.

Chapter 2 Fundamental Concepts

Chapter 2 has been shortened, with former Section 2.2 now included in a newly developed Chapter 6 that is devoted to algebraic thinking. Sections 2.1 and 2.3 from the fifth edition remain, with revisions in these sections focused on enhancing discussions of sets and numeration.

Section 2.1 gives students tools that enable them to talk about sets and subsets and to use Venn diagrams when the need arises in other chapters, such as to understand the relationship between different sets of numbers.

Section 2.2 includes the development of children's understanding of numeration and its historical development, both of which students find fascinating. Exploration 2.3 (Alphabitia) is one of the most powerful we have used. Most of our students report this to be the most significant learning and/or turning point in the semester. The exploration unlocks powerful understandings related to numeration, which the text supports by discussing the evolution of numeration systems over time and exploring different bases.

Chapter 3 The Four Fundamental Operations of Arithmetic

Portions of the fifth edition's Chapter 4, Number Theory, are now integrated into Chapter 3, as appropriate. For example, divisibility may now be found in Section 3.4, Understanding Division. Several discussions are now rewritten with more emphasis on place value and visual representations of numbers. The goals of Chapter 3 otherwise are the same. Students see how the concepts of the operations, coupled with an understanding of base ten, enable them to understand how and why procedures that they have performed by memorization for years actually work. In addition to making sense of standard algorithms, we present alternative algorithms in both the text and explorations. Our students have found these algorithms to be both enlightening and fascinating.

Chapter 4 Extending the Number System

The sets of integers, fractions, and decimals represent three historically significant extensions to the set of whole numbers. To enhance the discussion of fractions, Singaporean bar models are used. The concepts of least common multiple and greatest common divisor are integrated into the fraction section when needed for simplifying and for common denominators.

In Exploration 4.5 (Making Manipulatives), students construct fraction manipulatives and then look for rules when ordering fractions, a critically important first step in seeing fractions as more than numerator and denominator. In Exploration 4.19 (Meanings of Operations with Fractions), having students represent problem situations with diagrams requires them to adapt their understanding of the four operations to fraction situations. Having first constructed this concept through exploration, students can approach Investigation 4.2k (Ordering Rational Numbers) with a richer understanding of what it really means to say that one fraction is greater than another.

While Chapters 3 and 4 have been arranged according to the manner in which many instructors prefer this content to appear, it is not fixed. For those instructors who prefer a more "operations-centric" approach to the course, we offer an alternative organization of topics as follows:

Chapter 1 Foundations for Learning Mathematics

Chapter 2 Fundamental Concepts

4.1 Integers

4.2 Fractions and Rational Numbers

3.1 Understanding Addition

3.2 Understanding Subtraction

4.3 Understanding Operations with Fractions (first half addition and subtraction of fractions)

3.3 Understanding Multiplication

3.4 Understanding Division

4.3 Understanding Operations with Fractions (second half, multiplication and division of fractions)

4.4 Beyond Integers and Fractions

Chapter 5 Proportional Reasoning

Chapter 6 Algebraic Thinking, and so on

Chapter 5 Proportional Reasoning

The investigations and explorations in Chapter 5 are conceptually rich and provide many real-life examples so that students can enjoy developing an understanding of multiplicative relationships.

Chapter 6 Algebraic Thinking

In response to requests from reviewers, we have included a new chapter devoted to algebraic thinking. Chapter 6 explores patterns, the concept of a variable, and solving equations and inequalities using different models, including Singaporean bar models. The four sections are arranged under the National Council of Teachers of Mathematics (NCTM) algebra structure of understanding patterns, relations, and functions; representing and analyzing math situations and structures using algebraic symbols; using mathematical models to represent and understand quantitative relationships; and analyzing change in various contexts.

Chapter 7 Uncertainty: Data and Chance

In this chapter, students carefully walk through the stages of defining a question, collecting data, interpreting data, and then presenting data. We are particularly excited that the investigations with the concepts of mean and standard deviation remain successful with students. As a result, students can express these ideas conceptually instead of simply reporting the procedure.

Chapter 8 Geometry as Shape

In Chapter 8, you have the option of introducing geometry through explorations with tangrams, Geoboards, or pentominoes. This more concrete introduction allows students with unpleasant or failing memories of geometry to build confidence and understanding while engaging in rich mathematical explorations.

Chapter 9 Geometry as Measurement

This chapter addresses measurement from a conceptual framework (i.e., identify the attribute, determine a unit, and determine the amount in terms of a unit) and a historical perspective. Both the explorations and investigations get students to make sense of measurement procedures and to grapple with fundamental measurement ideas. Exploration 9.2 (How Tall?) generates many different solution paths and ideas and many discussions about indirect measurement and precision. Exploration 9.5 (What Does π Mean?) has demystified π in the minds of many students and is a wonderful exercise in communication. Exploration 9.11 (Irregular Areas) requires students to apply notions of measuring area to a novel situation. Students will hypothesize many different strategies, some of which are valid and some of which are not. The text looks at the larger notion of measurement, presents the major formulas in a helpful way, and illustrates different problem-solving paths.

Some of the most significant revisions to this chapter have been made to increase conceptual understanding of the concepts of measurement such as perimeter, area, and volume.

Chapter 10 Geometry as Transforming Shapes

The geometric transformations that we explore in Chapter 10 can be some of the most interesting and exciting topics of the course. Quilts and tessellations both spark lots of interest and provoke good mathematical thinking. The text develops concepts and introduces terms that help students to refine understanding that emerges from explorations.

PREFACE

Owning versus Renting

This course is about developing and *retaining* the mathematical knowledge that students will need as beginning mathematics teachers. We prefer to say that we are going to *uncover* the material rather than *cover* the material. The analogy to archaeology is useful. When archaeologists explore a site, they carefully *uncover* the site. As time goes on, they see more and more of the underlying structure. This is exactly what can and should happen in a mathematics course. When this happens, students are more likely to *own* rather than to *rent* the knowledge.

There are three ways in which this textbook supports owning versus renting:

1. Knowledge is constructed.
2. Connections are reinforced.
3. Problems appear in authentic contexts.

1. Constructing Knowledge

When students are given problems, such as appear in Investigations throughout the textbook, that involve them in grappling with important mathematical ideas, they learn those ideas more deeply than if they are simply presented with the concepts via lecture and then are given problems for practice. Additionally, there is a need to shift the focus from students studying mathematics to students doing mathematics. That is, students are looking for patterns, making and testing predictions, making their own representations of a problem, inventing their own language and notation, etc.

Investigation 1.2d (Pigs and Chickens) confronts a common misconception—that there is one right way to solve math problems—by exploring five valid solution paths to the problem. This notion of multiple solution paths is an important part of the book.

INVESTIGATION 1.2d Pigs and Chickens

A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter, "I count 24 heads and 80 feet. How many pigs and how many chickens are out there?"

Before reading ahead, work on the problem yourself or, better yet, with someone else. Close the book or cover the solution paths while you work on the problem.

Compare your answer to the solution paths below.

DISCUSSION

STRATEGY 1 Use random trial and error

One way to solve the problem might look like what you see in Figure 1.3.

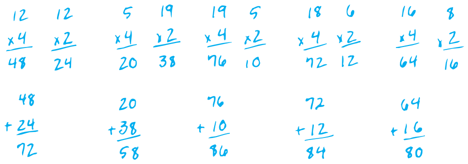


Figure 1.3

2. Reinforcing Connections

Understanding can be defined in terms of connections; that is, the extent to which you *understand* a new idea can be seen by the *quality* and *quantity* of connections between that idea and what you already know. There are two ways in which connections are built into the structure of the text.

1. Mathematical connections

Owning mathematical knowledge involves connecting new ideas to ideas previously learned. It also involves truly understanding mathematics, not just memorizing formulas and definitions.

- **CONNECTIONS AMONG CONCEPTS ARE EMPHASIZED**

Investigation 1.1d helps students see how the algebraic formula is closely connected to guess–check–revise. Investigation 1.2i is later connected both to fractions and to remainder. In Chapter 3, the four operations are constantly connected to each other in their development. Then in Chapter 4, the connections between operations with fractions and operations with whole numbers are discussed, as are how decimals are connected to whole numbers and to fractions. In Chapter 5, we look back at some problems done in Chapter 4 and see how they can now be solved more efficiently with the concepts of ratio. In Chapter 10, students see how our work with numbers and shapes is similar.

- **THE *HOW* IS CONNECTED TO *WHY***

In this way, students know not only how the procedure works but also why it works. For example, students understand why we move over when we multiply the second row in whole number multiplication; they realize that "carrying" and "borrowing" essentially equate to trading tens for ones or ones for tens; they understand why we first find a common denominator when adding fractions; and they see that π is how many times you can wrap any diameter around the circle.

2. Connections to children's thinking

In this book you will see a strong focus on children's thinking, for two reasons. First, much work with teachers focuses on the importance of listening to the students' thinking as an essential part of good teaching. If students experience this in a math course, then by the time they start teaching, it is part of how they view teaching. Second, when students see examples of children's thinking and see connections between problems in this course and problems children solve, both the quality and quantity of the students' cognitive effort increase.

3. Authentic Problems

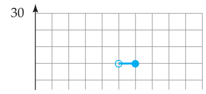
Although most texts have many "real-life" problems, this text differs in how those problems are made and presented.

In Section 6.3, the question of paying a baby-sitter is explored. This situation is often portrayed as a linear function: for example, if the rate is \$10 per hour, $y = 10x$. However, in actuality, it is not a linear function but rather a stepwise function.

Figure 6.3

MATHEMATICS

The graph that represents this baby-sitting function is often confusing to students who see it for the first time. However, it is relatively common in real-world mathematics and is a member of the subset of functions called step functions.



A closer look at paying the baby-sitter At first glance, the question of how much to pay the baby-sitter is simple: Multiply the hours sat by 8. However, let us use the problem-solving strategy "act it out" to examine this problem more closely. For example, what if Ellen baby-sat from 7 to 11:15? How much would you pay her? Think before reading on. . . .

Some people say \$32. Some people say \$36—they round up to the nearest half-hour. In actuality, different people have different ways of determining how much to pay a baby-sitter. Let us examine the case of a couple, who rounds up the time to the nearest half-hour. We could now represent their process for paying the baby-sitter in each of the ways we have just examined. Is the relationship between time sat and dollars earned a functional relationship in the case of the couple? Think and then read on. . . .

If you were to graph this relationship, what would the graph look like? Try to make your own graph before reading on. . . .

Similarly, when determining the cost of carpeting a room, the solution path is often presented as dividing the area of the room by the cost per square yard; again, this is not how the cost is actually determined.

In this book, you will find many problems—problems we have needed to solve, problems friends have had, problems children have had, problems we have read about—where the content fits with the content of this course.

You will find problems in the text where students are asked to state the assumptions they make in order to solve the problem (e.g., Section 5.2, Exercises 33 to 38). You also will find problems that have the messiness of "real-life" problems, where the problem statement is ambiguous, too little or too much information is given, or the information provided is contradictory.

Problems 33–38 require you to make some assumptions in order to determine an answer. Describe and justify the assumptions you make in determining your answer.

33. Let's say that you read in the newspaper that last year's rate of inflation was 7.2%.

- **a.** If your grocery bill averaged \$325 per month last year, about how much would you expect your grocery bill to be this year?
- **b.** Let's say you received a \$1200 raise, from \$23,400 per year to \$24,600 per year. Did your raise keep you ahead of the game, or are you falling behind?

34. There was a proposal in New Hampshire in 1991 to reduce the definition of "drunk driving" from an alcohol blood content of 0.1 to 0.08. Explain why some might consider this a little drop and others might consider it a big drop. What do you think?

35. Which would you prefer to see on a sale sign at a store: \$10 off or 10% off? Explain your choice.

36. ***Classroom Connection*** Refer to Investigation 5.2b. Jane still doesn't understand the problem. Roberto tries to help her make sense of the problem by saying that the 8% means that if we were to select 100 students at the college, 8 of them would be working full-time. What do you think?

37. Annie has just received a 5% raise from her current wage of \$9.80 per hour.

- **a.** What is her new wage?
- **b.** What would this amount to over a year?
- **c.** What assumptions did you make in order to answer part (b)?
- **d.** What if the raise had been 5.4%?

Features

What do you think?

What-do-you-think questions appear at the start of each section to help students focus on key ideas or concepts that appear within the sections.

6.1 Understanding Patterns, Relations, and Functions

What do you think?

- How are patterns related to algebraic thinking?
- What are some examples of functions in everyday life?
- What is a reason for developing algebraic thinking in elementary school?

Investigations

Investigations are the primary means of instruction, uniquely designed to promote active thinking, reasoning, and construction of knowledge. Each investigation presents a problem statement or scenario that students work through, often to uncover a mathematical principle relevant to the content of the section. The "Discussion" that follows the problem statement provides a framework for insightful solution logic.

INVESTIGATION 3.1f Children's Mistakes

The problem below illustrates a common mistake made by many children as they learn to add. Understanding how a child might make that mistake and then going back to look at what lack of knowledge of place value, of the operation, or of properties of that operation contributed to this mistake is useful. What error on the part of the child might have resulted in this wrong answer?

The problem: $38 + 4 = 78$

DISCUSSION

In this case, it is likely that the child lined up the numbers incorrectly:

$$\begin{array}{r} 4 \\ +\ 38 \\ \hline 78 \end{array}$$

Giving other problems where the addends do not all have the same number of places will almost surely result in the wrong answer. For example, given $45 + 3$, this child would likely get the answer 75. Given $234 + 42$, the child would likely get 654. In this case, the child has not "owned" the notion of place value. Probably, part of the difficulty is not knowing expanded form (for example, that 38 means $30 + 8$—that is, 3 tens and 8 ones). An important concept here is that we need to add ones to ones, tens to tens, etc. Base ten blocks provide an excellent visual for this concept as students can literally see why they cannot add 4 ones to 3 tens.

CLASSROOM CONNECTION

A friend of mine, David Sobel, was talking about mathematics with his six-year-old daughter, Tara. David had just shown Tara that $20 + 20 = 40$. Tara thought for a moment and then proudly announced that $50 + 50$ must be 70. When David asked how she had got that answer, she said, "When you add the same numbers that have a zero at the end, you just skip ten!"

Questions in the Text

To encourage active learning outside of the Investigations, questions appear embedded within the text, often accompanied by the icon . These "thinking" questions require students to pause in their reading to reflect or to complete a short exercise before continuing. Answers to these questions can be found in Appendix B in the back of the textbook.

> Translate the following Babylonian numerals into our system. Check your answers in Appendix B.
>
> 
>
>
> Translate the following amounts into Babylonian numerals.
>
> **4.** 1202 **5.** 304

Classroom Connections

Connections to the Classroom, denoted with the icon , are found throughout the textbook. The boxed Connections that appear in the margins provide observations, tips, and notes about the elementary/middle-school classroom. Assignments from actual elementary/middle-school books appear throughout as well so that students can see how the material they are learning will directly apply. Connections are also found in the exercises that highlight children's work.

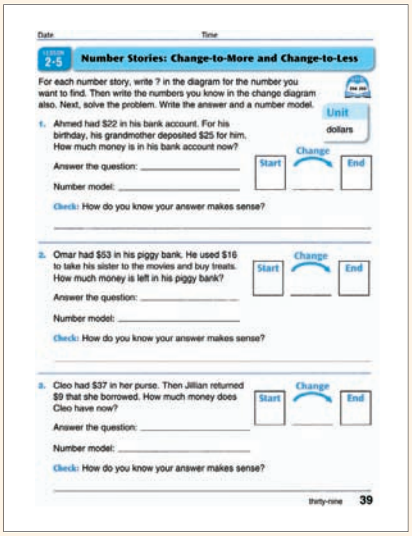
> **CLASSROOM CONNECTION**
>
> This question asking how two things are alike and how they are different is an important teaching structure and one that we will revisit over the course of the book. You may remember it in a common *Sesame Street* feature: Three of These Things Belong Together. For example, they might show a triangle, a square, a hexagon, and a circle. The answer is that the circle doesn't belong because it doesn't have line segments. We will examine this idea of asking how things are alike and how they are different throughout the textbook.

Section 3.2 Understanding Subtraction 101

CLASSROOM CONNECTION

Grade 3

Can you describe the context of addition or subtraction in each story?


Date Time

LESSON 2·5 **Number Stories: Change-to-More and Change-to-Less**

For each number story, write ? in the diagram for the number you want to find. Then write the numbers you know in the change diagram also. Next, solve the problem. Write the answer and a number model.

Unit: dollars

1. Ahmed had $22 in his bank account. For his birthday, his grandmother deposited $25 for him. How much money is in his bank account now?
   Answer the question: ____
   Number model: ____
   Start — Change — End
   Check: How do you know your answer makes sense?

2. Omar had $53 in his piggy bank. He used $16 to take his sister to the movies and buy treats. How much money is left in his piggy bank?
   Answer the question: ____
   Number model: ____
   Start — Change — End
   Check: How do you know your answer makes sense?

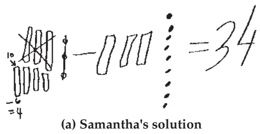
3. Cleo had $37 in her purse. Then Jillian returned $9 that she borrowed. How much money does Cleo have now?
   Answer the question: ____
   Number model: ____
   Start — Change — End
   Check: How do you know your answer makes sense?

thirty-nine 39

From *Everyday Mathematics, Grade 3:* The University of Chicago School Mathematics Project: Student Math Journal, Volume 1, by Max Bell et al., Lesson 2-5, p. 39. Reprinted by permission of The McGraw-Hill Companies, Inc.

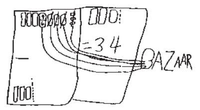
11. ***Classroom Connection*** Following are two children's solutions to 73 − 39, taken from the December 2001 issue of *Teaching Children Mathematics* (p. 231). Study each child's work, and then describe how that child would find the answer to 65 − 28. Then explain why that method works.

a.



(a) Samantha's solution

b.



(b) Alice's solution

Source: *Teaching Children Mathematics*, by the National Council of Teachers of Mathematics.

Margin Notes

To help round out the mathematics education of pre-service teachers, other special margin notes are provided.

▼ ***Outside the Classroom*** boxes highlight applications and uses of mathematical concepts and procedures in the business world, science, and everyday life.

OUTSIDE THE CLASSROOM

Skateboarders and snowboarders use reflex angles to describe some of their moves. What do you think they mean when they talk about doing a frontside 540 or a backside 720?

LANGUAGE

The term *ratio* comes from the Latin verb *ratus,* which means "to think or estimate." Many mathematicians in the sixteenth and seventeenth centuries used the word *proportion* for ratio. Even today you hear the two terms used interchangeably; for example, instructions for making a certain color might say, "Mix the two colors in the following proportion—3 : 2."

◀ ***Language*** boxes include the etymology of selected terms and/or describe nuances of terms.

HISTORY

Much of the mathematical notation we use is actually quite recent. The symbols for addition and subtraction, + and −, first appeared in Germany in the late 1400s. These symbols were first used to indicate sacks that were surplus or minus in weight. In 1631, William Oughtred first used the letter x to represent multiplication. Italian merchants introduced the symbol for division (÷) in the 1400s to indicate a half. For example, they wrote 4 ÷ to indicate $4\frac{1}{2}$. The equals sign first appeared in the late 1500s in a book by Robert Recorde.

◀ ***History*** boxes present interesting side notes, relevant to concepts developed in the text.

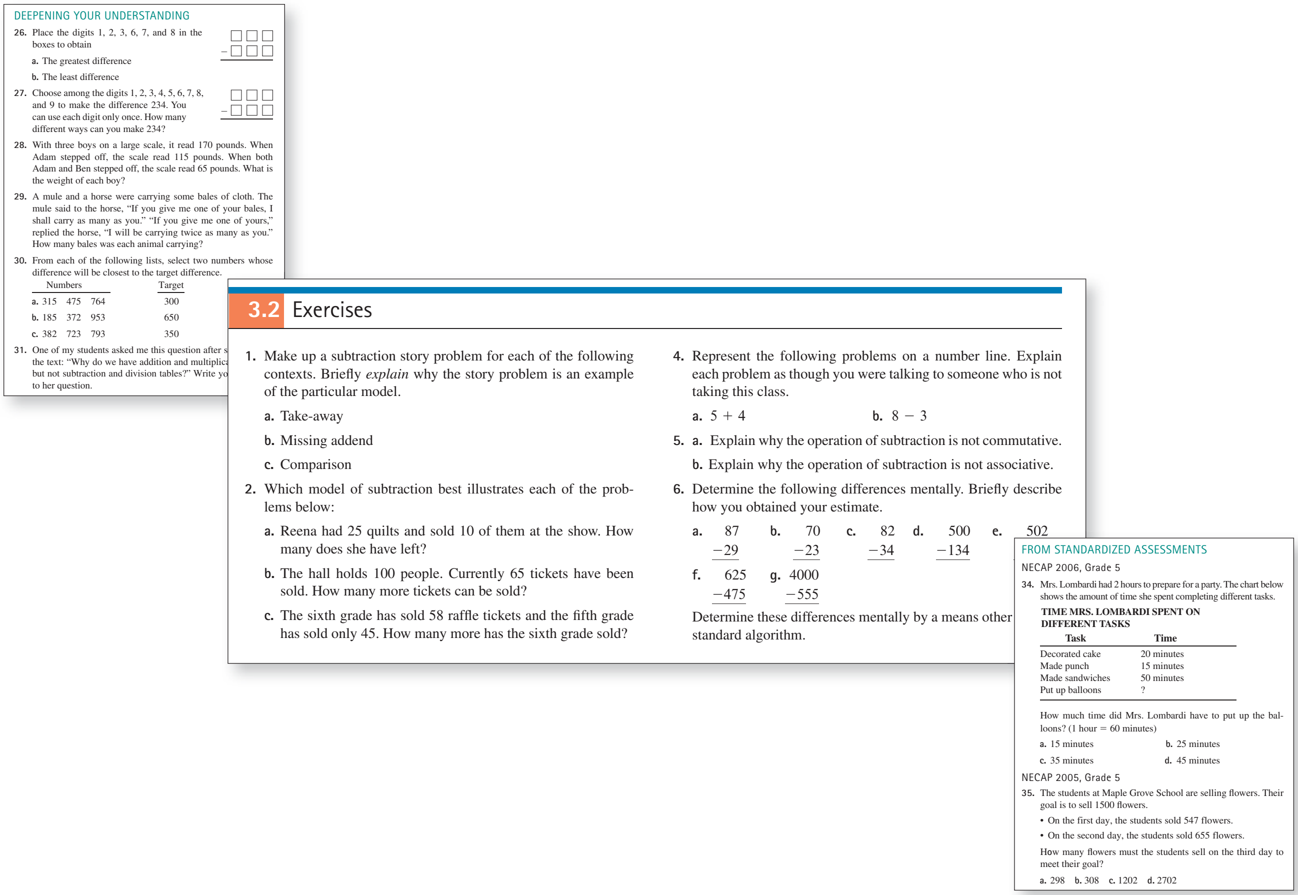
▼ ***Mathematics*** boxes associate a previously discussed concept or other related math idea with the topic under discussion.

MATHEMATICS

If you did Exploration 2.3, recall how strange the Alphabitian system was to you. What observations made you more comfortable with adding? Can you think of analogous observations that might make the learning of base ten addition facts easier for young children?

Section Exercises

The exercises are designed to give students a deeper sense and awareness of the kinds of problems their future students are expected to solve at various grade levels, as well as to increase their own proficiency with the content. Special subcategories appear toward the end of each set of exercises. *Deepening Your Understanding* exercises go a step beyond, encouraging students to extend their thinking beyond the basics. *From Standardized Assessments* exercises derive from exams such as the NECAP and NAEP to give students a sense of the types of questions found on diverse national exams at various grade levels. Questions are also included from the Smarter Balanced Assessment Consortium which is developing assessments for Common Core State Standards.



DEEPENING YOUR UNDERSTANDING

26. Place the digits 1, 2, 3, 6, 7, and 8 in the boxes to obtain

a. The greatest difference

b. The least difference

27. Choose among the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 to make the difference 234. You can use each digit only once. How many different ways can you make 234?

28. With three boys on a large scale, it read 170 pounds. When Adam stepped off, the scale read 115 pounds. When both Adam and Ben stepped off, the scale read 65 pounds. What is the weight of each boy?

29. A mule and a horse were carrying some bales of cloth. The mule said to the horse, "If you give me one of your bales, I shall carry as many as you." "If you give me one of yours," replied the horse, "I will be carrying twice as many as you." How many bales was each animal carrying?

30. From each of the following lists, select two numbers whose difference will be closest to the target difference.

Numbers			Target
a. 315	475	764	300
b. 185	372	953	650
c. 382	723	793	350

31. One of my students asked me this question after s the text: "Why do we have addition and multiplica but not subtraction and division tables?" Write yo to her question.

3.2 Exercises

1. Make up a subtraction story problem for each of the following contexts. Briefly *explain* why the story problem is an example of the particular model.

a. Take-away

b. Missing addend

c. Comparison

2. Which model of subtraction best illustrates each of the problems below:

a. Reena had 25 quilts and sold 10 of them at the show. How many does she have left?

b. The hall holds 100 people. Currently 65 tickets have been sold. How many more tickets can be sold?

c. The sixth grade has sold 58 raffle tickets and the fifth grade has sold only 45. How many more has the sixth grade sold?

4. Represent the following problems on a number line. Explain each problem as though you were talking to someone who is not taking this class.

a. $5 + 4$ **b.** $8 - 3$

5. **a.** Explain why the operation of subtraction is not commutative.

b. Explain why the operation of subtraction is not associative.

6. Determine the following differences mentally. Briefly describe how you obtained your estimate.

a. $87 - 29$ **b.** $70 - 23$ **c.** $82 - 34$ **d.** $500 - 134$ **e.** 502

f. $625 - 475$ **g.** $4000 - 555$

Determine these differences mentally by a means other standard algorithm.

FROM STANDARDIZED ASSESSMENTS

NECAP 2006, Grade 5

34. Mrs. Lombardi had 2 hours to prepare for a party. The chart below shows the amount of time she spent completing different tasks.

TIME MRS. LOMBARDI SPENT ON DIFFERENT TASKS

Task	Time
Decorated cake	20 minutes
Made punch	15 minutes
Made sandwiches	50 minutes
Put up balloons	?

How much time did Mrs. Lombardi have to put up the balloons? (1 hour = 60 minutes)

a. 15 minutes **b.** 25 minutes

c. 35 minutes **d.** 45 minutes

NECAP 2005, Grade 5

35. The students at Maple Grove School are selling flowers. Their goal is to sell 1500 flowers.

- On the first day, the students sold 547 flowers.
- On the second day, the students sold 655 flowers.

How many flowers must the students sell on the third day to meet their goal?

a. 298 **b.** 308 **c.** 1202 **d.** 2702

Section Summary

Each section ends with a summary that reviews the main ideas and important concepts discussed.

SUMMARY 3.2

We have now examined addition and subtraction rather carefully. In what ways do you see similarities between the two operations? In what ways do you see differences? Think and then read on. . . .

One way in which the two processes are alike is illustrated with the part–whole diagram used to describe each operation. These representations help us to see connections between addition and subtraction. In one sense, addition consists of adding two parts to make a whole. In one sense, subtraction consists of having a whole and a part and needing to find the value of the other part.

We see another similarity between the two operations when we watch children develop methods for subtraction; it involves the "missing addend" concept. That is, the problem $c - a$ can be seen as $a + ? = c$.

We saw a related similarity in children's strategies. Just as some children add large numbers by "adding up," some children subtract larger numbers by "subtracting down."

Earlier in this section, subtraction was formally defined as $c - b = a$ if $a + b = c$. The negative numbers strategy that some children invent brings us to another way of defining subtraction, which we will examine further in Chapter 4 when we examine negative numbers. That is, we can define subtraction as adding the inverse: $a - b = a + -b$.

A very important way in which the two operations are different is that the commutative and associative properties hold for addition but not for subtraction.

Looking Back

Each chapter concludes with *Looking Back*—a study tool that brings together all the important points from the chapter. *Looking Back* includes *Questions to Summarize Big Ideas* (NEW!), which ask students to reflect on the main ideas from the chapter; *Chapter Summary*, which lists major take-aways and terminology from the chapter; and *Review Exercises*, which provide an opportunity for students to put concepts from the chapter into practice.

LOOKING BACK on chapter three

QUESTIONS TO SUMMARIZE BIG IDEAS

1. What are some of the different models for addition, subtraction, multiplication, and division?
2. How can you use base ten blocks to model the algorithms for each of the operations?
3. How are these models similar and different in a base other than ten?
4. Which algorithms for the operations are different from what you learned in elementary school?
5. What are the tools for determining divisibility and why do they work?
6. Look back at the Mathematical Practices of the Common Core State Standards. In what ways did you engage in those practices during this chapter?
7. What parts of this chapter are less clear to you at

CHAPTER 3 SUMMARY

1. Many students have said that really understanding base ten and the four operations, was, for them, the beginning of a new attitude toward mathematics. We will continue to examine new and important mathematical ideas throughout this book, but the foundation for much of elementary mathematics has now been laid.
2. Each operation has multiple meanings.
3. Many algorithms have been developed to enable us to compute more efficiently.
4. The standard algorithm for each operation does not connect equally well to each meaning of the operation.
5. Being able to make sense of algorithms requires:
   - The ability to apply base ten and place value concepts
   - The ability to compose and decompose the numbers (for example, to use expanded form)
6. Patterns enable us to understand the operations more deeply.
7. In many real-life problems, the answer depends on knowing how to interpret one's computation.
8. Being able to perform mental math and to estimate requires
   - The ability to apply base ten and place value concepts
   - The ability to compose and decompose the numbers (for example, to use expanded form)
   - The ability to apply properties of the operations, especially the commutative, associative, and distributive properties
9. Numbers in real-life settings are sometimes exact, sometimes rounded, and sometimes estimates.
10. In real-life problem-solving, one needs to know when to find an exact answer and when to find an estimate.
11. Real-life problem solvers need to know whether their estimates are reasonable.
12. People may use rounded numbers rather than exact numbers for a variety of reasons.

BASIC CONCEPTS

Section 3.1: Understanding Addition

Addition terminology:

addition **78** sum **78**
addends **78**

Addition contexts:

discrete **76** continuous, measured **76**

Addition models:

pictorial **76** number line **78**
tables **79**

REVIEW EXERCISES chapter three

1. State the problem that is represented in each case below:

Explorations

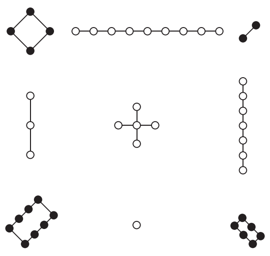
The icon that appears throughout the text references additional activities that may be found in the *Explorations Manual*. Explorations present new ideas and concepts for students to engage with "hands on."

Mathematics for Elementary School Teachers p. 27

EXPLORATION 1.6 Magic Squares

Magic squares have fascinated human beings for thousands of years. The oldest recorded magic square, the Lo Shu magic square, dates to 2200 B.C. and is supposed to have been marked on the back of a divine tortoise that appeared before Emperor Yu when he was standing on the bank of the Yellow River. In the Middle Ages, many people believed magic squares would protect them against illness! Even in the twenty-first century, people in some countries still use magic squares as amulets.

As a teacher, you will find that many of your students love working with magic squares and other magic figures.



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Foundations for Learning Mathematics

1

Knowing mathematics means being able to use it in purposeful ways. To learn mathematics, students must be engaged in exploring, conjecturing, and thinking rather than only in rote learning of rules and procedures. Mathematics learning is not a spectator sport. When students construct personal knowledge derived from meaningful experiences, they are much more likely to retain and use what they have learned. This fact underlies [the] teacher's new role in providing experiences that help students make sense of mathematics, to view and use it as a tool for reasoning and problem solving.[1]

—National Council of Teachers of Mathematics

SECTION 1.1 Getting Started and Problem Solving

What do you think?

- Respond to the prompt: Mathematics is ____________.
- Describe a few of your experiences learning mathematics as an elementary school student.
- Describe your attitudes toward mathematics and where you think these attitudes come from.
- What attitudes do you have about taking this course?

You are at the beginning of a course where you will re-examine elementary school mathematics to understand these concepts on a much deeper level, and to learn why the mathematical procedures and formulas actually work. On this journey, you will learn several ways to see and think about concepts and procedures that you may have previously simply memorized. This deeper understanding will lead to increased confidence and comfort level with mathematics. Your approach to this course, and to teaching mathematics, depends on the attitudes and beliefs you bring to the classroom; in subtle and not so subtle ways, you may pass these beliefs along when you enter the classroom as a teacher. Reflect on how you answered the questions above. Whatever your feelings about mathematics, consider where these feelings come from. Research suggests that people who have mathematics anxiety can relate it back to a teacher and/or experience in their elementary or middle school years. Think about the best math teacher you have had as well as the worst math teacher you have had. Consider the skills and qualities that each of them had that led to your experience of them. What skills and qualities do you have and need to further develop to become an excellent math teacher?

BELIEFS AND ATTITUDES ABOUT MATHEMATICS

This preliminary exercise is designed with two purposes in mind. First, it will help you examine and reflect on your beliefs and attitudes at the beginning of the course. Second, it will help you see a practical use of mathematics.

Rate your attitudes

Seven pairs of statements concerning attitudes and beliefs about mathematics are given in Table 1.1. Score your beliefs in the following manner:

- If you strongly agree with the statement in column A, record a 1.
- If you agree with the statement in column A more than with the statement in column B, record a 2.
- If you agree with the statement in column B more than with the statement in column A, record a 3.
- If you strongly agree with the statement in column B, record a 4.

Adaptive and maladaptive beliefs

Before we discuss your responses to Table 1.1 let us examine attitudes. I have worked with thousands of students during my time as a teacher, and I know from experience and from reading research that one's beliefs about mathematics can influence how one learns and teaches.

TABLE 1.1

Column A		Column B
1. There will be many problems in this book that I won't be able to solve, even if I try really hard.	1 2 3 4	**1.** I believe that if I try really hard, I can solve virtually every problem in this book.
2. There is only one way to solve most "word" problems.	1 2 3 4	**2.** There is usually more than one way to solve most "word" problems.
3. The best way to learn is to memorize the different kinds of problems—rate problems, mixture problems, coin problems, etc.—and how to solve them.	1 2 3 4	**3.** The best way to learn is to make sure that I understand each step.
4. Some people have mathematical minds and some don't. Nothing they do can *really* make a difference.	1 2 3 4	**4.** Some students may have more aptitude for mathematics than others, but everyone can become competent in mathematics.
5. The teacher's job is to show us how to do problems and then give us similar problems to practice.	1 2 3 4	**5.** The teacher's job is more like that of a coach or guide—to help us develop the problem-solving tools we need.
6. A good test consists of problems that are just like the ones we have done in class.	1 2 3 4	**6.** A good test has problems at a variety of levels of difficulty, including some that are not just like the ones in the book.
7. I don't need to know all the ideas covered in this book because I'm going to teach younger children.	1 2 3 4	**7.** Even teachers of young children need to have a good understanding of the ideas in this book.
	Total ______	

OUTSIDE THE CLASSROOM[2]

The pervasiveness of negative attitudes toward mathematics was powerfully illustrated in 1992 when Mattel introduced a new talking Barbie doll that said, "Math is tough." Now this may be true for some people, but having Barbie say it only reinforced that stereotypical perception of mathematics in the United States, especially among females. Mattel was persuaded to change Barbie's statement.

Some students have **adaptive beliefs** that help them approach math with a positive and confident attitude. Some have **maladaptive beliefs** that keep them from thinking of learning as an evolving and enjoyable process.

In Table 1.1, the statements in column A indicate maladaptive beliefs, and the statements in column B indicate the corresponding adaptive beliefs. If you take the arithmetic average, or *mean,* of your scores (by adding up your scores and dividing by 7), you will get a number that we could call your belief index. If your belief index is low and you encounter difficulties in this course, it may be because some of your beliefs are hindering your ability to learn the material. If you find this course frustrating, try to discuss your beliefs with your professor, with someone at a math center (if your college has one), or with a friend who is doing well in the course. Deepening your understanding of mathematics through this course and beyond will help you to have a positive attitude about mathematics. Approach this course with an open mind toward learning and a belief that everyone (including yourself) can understand mathematics. Your future students are depending on you to deepen your understanding of math and to have positive attitudes about it.

LANGUAGE

Whenever you see the pencil icon, stop and think and briefly write your thoughts before reading on. Students who take the time to think and write after these points (or at least to pause and think) say that it makes a big difference in how much they learn.

WHAT IS MATHEMATICS?

What is mathematics? Think about this question for a minute and then read on. . . .

You may be surprised to learn that not all mathematicians give the same response to this question. *On the Shoulders of Giants: New Approaches to Numeracy*[3] was written partly to help expand people's views of mathematics beyond the common stereotype of "mathematics is a bunch of formulas and rules for numbers." A group of mathematicians and mathematics educators brainstormed a number of possible themes for that book. In the end, it was agreed that the idea of *pattern* permeates all fields of mathematics. Five mathematicians were asked to write chapters on the following themes:

Dimension. In school, you have studied two- and three-dimensional shapes. Mathematicians have gone far beyond three dimensions for years. Recently, a field of mathematics has opened up the exploration of fractional dimensions. For example, the coastline of Britain (which can be modeled by a long, squiggly line) has been calculated to have a dimension of 1.26.

Quantity. This begins (with children) with the question "how many," for which the counting numbers (1, 2, 3, . . .) are appropriate; it moves in complexity to the question "how much," for which fractions and decimals were invented, and then to questions far more complex, for which other numbers and systems were invented.

MATHEMATICS

Keith Devlin has written several fascinating and readable books on this subject,[4] one of which is a companion to a PBS series entitled *Life by the Numbers* (your college or local library probably has this book). The chapter titles for *Mathematics: The Science of Patterns* are "Counting," "Reasoning and Communicating," "Motion and Change," "Shape, Symmetry and Regularity," and "Position." Devlin discusses (among many other things) how mathematicians helped us to understand why leopards have spots and tigers have stripes, how mathematicians helped American ice skaters learn how to perform triple axel jumps, and how we use mathematics to measure the heights of mountains.

Uncertainty. Questions of uncertainty permeate everyday life: How long will I live? What are my chances of getting a job after I graduate? What are the chances that my baby will be "normal"?

Shape. Humans' relationship with shape has a fascinating history—the shape of one's environment (desert, forest, mountain), what shape is best for packaging, the shapes that artists make, and the shapes we manufacture for quilts, clothing, and so on.

Change. We live in a world that is constantly changing. The development of computers enables us better to understand and manage change, whether it be the changing climate, the change in epidemics (such as AIDS), the change in populations (human and animal), or changes in the economy.

Mathematics is far more than titles of courses and chapters in textbooks—whole numbers, fractions, decimals, percents, algebra, geometry, etc. The topics in textbooks represent tools that are needed in order to answer important questions about dimension, quantity, uncertainty, shape, and change.

> The numbers, lines, angles, shapes, dimensions, averages, probabilities, ratios, operations, cycles, and correlations that make up the world of mathematics enable people to make sense of a universe that otherwise might seem to be hopelessly complicated.[5]

CLASSROOM CONNECTION

Deborah Meier, an award-winning principal, urged her teachers to think critically and often[6] asked them, "So what?" I encourage this of you, too. There are many wonderful teachers in this country, but there are also many teachers for whom mathematics is just another subject to be "covered." Do you want your students to form this view of mathematics and carry it from your classroom? If not, then you need to ask yourself the "So what?" question frequently, and when you cannot give a satisfactory answer, talk with your instructor or go elsewhere—to the Web, to articles in *Teaching Children Mathematics,* to classrooms—and see for yourself the kinds of mathematical experiences that we want young children to have in school.

Mathematics is both beauty and truth. Two plus two always equals four. The distance around a circle is always a little more than three (actually pi) times the distance across the circle. Throughout this text, we hope you will appreciate more and more of the beautiful truths of mathematics.

USING MATHEMATICS

Let us now turn our attention to how people use mathematics in everyday situations and in work situations. Take a few minutes to jot down some instances in which you have used mathematics in your life and some instances in which you know that mathematics is used in different careers and work situations. Then read on. . . .

People use mathematics for various purposes, for example:

- To persuade a boss that our idea will make money
- To persuade a potential customer that our product will save money
- To predict—tomorrow's weather or who will win the election
- To make a personal decision—whether we can afford to buy a house
- To make a business decision—how much to charge for a new product or whether a new medicine (for example, a cure for AIDS) really works
- To help us understand how the world works (for example, why leopards have spots and tigers have stripes)
- To relax—many people enjoy Sudoku and other math puzzles

Solving problems

In each of these examples, people are using mathematics as a tool for solving problems. To decide whether you can afford a new car, you have to collect data (on insurance, for example), add decimals, and work with percents (such as sales tax and interest on the loan). The mathematics you use will help you solve the problem of whether to buy a car.

Think about this situation involving weather forecasters. In 1994 a hurricane brought severe rains to Georgia. Forecasters predicted that the Flint River would crest at 20 feet above flood level; the river actually crested at 13 feet above flood level, much to the relief of many residents. The forecasters used decimals, volume formulas, and conversions to determine the maximum volume of water that would be flowing. They also used computer models of flooding rivers, and the computer models were based on data collected on previous flooding.

Now that we have discussed mathematics in general, let's focus on the specialized mathematical understanding that teachers need.

MATHEMATICAL KNOWLEDGE FOR TEACHING

INVESTIGATION 1.1a More Than One Way to Multiply?

First, multiply 49 × 25 using any method you choose. Then consider how the following students solved the problem.

What method do you think each student is using in each solution below? Do you think each method will work for any two whole numbers?

Student A:	Student B:
$\begin{array}{r} 49 \\ \times 25 \\ \hline 245 \\ 98 \\ \hline 1225 \end{array}$	$\begin{array}{r} 49 \\ \times 25 \\ \hline 45 \\ 200 \\ 180 \\ 800 \\ \hline 1225 \end{array}$

DISCUSSION

We will look much more deeply into multiplication later in this course. For now, we use this example to illustrate how part of the role of a teacher is to be able to first understand that there are different methods for solving problems, and then be able to understand different strategies, and to know what to do with these different methods to develop deeper understanding. This is the mathematical understanding that is important for elementary teachers, and this deeper understanding is the focus of this course.

Student A used the method that many of us learned in elementary school of 5×49 and 20×49. Some of us learned to write the second row in the multiplication as 980, some were taught to move over a space as is shown here. Why do we do this? Because we are multiplying 25×40, not 25×4, we write it as 980, or as 98 with a space in the ones place.

We can use the properties and split the numbers up into their parts to figure this out.

$$49 \times 25 = 49 \times (20 + 5) = 49 \times 20 + 49 \times 5 = 980 + 245 = 245 + 980 = 1225$$

Student B used a method that is sometimes called partial products of $9 \times 5 = 45$, $40 \times 5 = 200$, $9 \times 20 = 180$, and $40 \times 20 = 800$. Another way to write this, which uses the distributive property more clearly is:

$$49 \times 25 = (40 + 9) \times (20 + 5) = 40 \times 20 + 40 \times 5 + 9 \times 20 + 9 \times 5 =$$
$$600 + 200 + 180 + 45 = 1225$$

The methods of both students are valid. One of the major ideas in this text is that there are multiple ways (often called "solution paths") to get to an answer. There is no ONE "right" way to solve any math problem. This may be different from what you have always thought about mathematics. You may have even had teachers who marked you "wrong" if you did not solve it their way.

As a teacher of elementary school mathematics, a deep understanding of mathematics will enable you to respond to the above type of situation that arises in an elementary school classroom. Simply being able to get the right answer is not sufficient. Teachers need a specialized understanding of mathematics that is flexible, connected, and conceptual. This course will help you develop that.

Teachers use mathematics every day in the classroom, but in different ways than others. In 1986, Lee Shulman used the term "pedagogical content knowledge" to refer to this specialized understanding of mathematics, which includes an understanding of multiple representations and examples, plus an understanding of what ideas may be more difficult for students and why these ideas are more difficult. In 2008, Deborah Loewenberg Ball, Mark Thames, and Geoffrey Phelps developed the framework depicted in Figure 1.1 showing different types of knowledge. This book is focused on the specialized content knowledge, which includes understanding multiple representations and multiple student procedures, and analyzing student errors. It is the type of math knowledge that teachers draw on every day.

CLASSROOM CONNECTION

Many students have told me that before they took this course, they thought there was just *one right way* to do a problem, and so they never looked for patterns but instead looked for formulas or procedures. Once the students started looking for patterns, they found them everywhere, and over time they learned how to use their awareness of patterns more powerfully. We can refer to different ways to solve a problem as different **solution paths**.

Explorations Manual 1.2

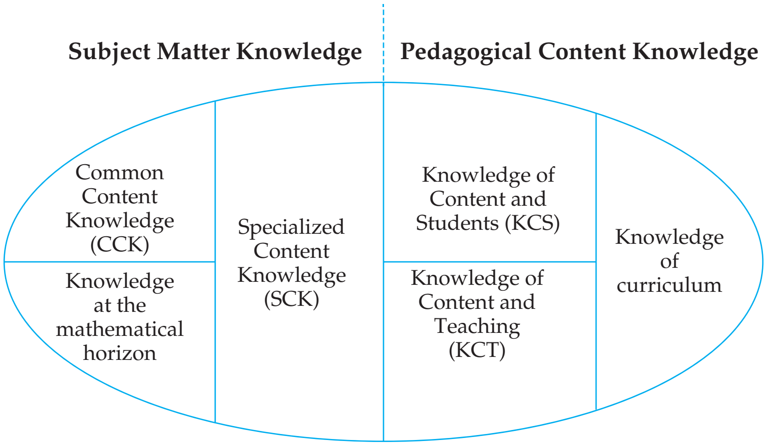

Figure 1.1
Source: Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372-400.

INVESTIGATION 1.1b Understanding Students' Errors

How do you think the answer was produced? What do you think the student was thinking that led to this error?

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 245 \\ 98 \\ \hline 343 \end{array}$$

How does this type of scenario draw upon specialized content knowledge?

DISCUSSION

Analyzing student errors draws upon a specialized understanding of mathematics in that the teacher needs to understand the mathematics deeply in order to identify the error and then to help the student to correct the misunderstanding.

This student does not understand that in the second step, we are actually multiplying 20 times 49 so it should be 980, not 98.

While most people could get a correct answer for 49×25, teachers need to understand the concept much more deeply. Many of us experienced elementary school mathematics as a series of procedures to memorize (like multiplying 49×25). In order to teach true understanding of mathematics, teachers must develop this specialized content knowledge.

LEARNING STYLES

Consider how you best learn, particularly mathematics. Do you prefer visuals? Do you need to experience it by "getting your hands on it"? Do you need to experience it through sound? Do you prefer to talk with others while solving a problem, or think about it alone? There are several theories about how people learn and think differently, including the Kolb learning styles, VAK (Visual Auditory Kinesthetic) learning styles, and multiple intelligences theory. As you go through this course, it may be helpful to think about how you best learn mathematics. There are several online surveys that can help you determine your own learning styles. You can locate free surveys upon searching the Internet for "learning styles surveys"; once you understand your own learning styles more deeply, you can adjust your learning experiences to accommodate them.

LANGUAGE

Many authors define understanding in terms of connections. That is, you truly understand an idea only if it is well-connected to other ideas, and your depth of understanding is connected to how many connections you are making and the quality of those connections.

PROBLEM SOLVING AND TOOLBOXES

A useful metaphor for problem solving is a **toolbox**. Imagine that your car breaks down and is towed to the garage, where a novice mechanic is the first to look at it. The novice will probably try a few standard procedures: Insert the key to see what happens, check the battery connections, look for a loose wire, and so on. If none of those strategies work, the novice mechanic will be stumped and will have to summon the senior mechanic. The senior mechanic may try the same basic procedures and may solve the problem by *interpreting* the results more skillfully. As you go through this course, you will learn new tools and how to use tools you already have more skillfully.

INVESTIGATION 1.1c Real-Life Problem Solving

Consider a few problems you have had in your life, and not necessarily math problems.

What steps did you take to solve these problems?

Use this recollection to make a list of general steps that you take to solve a problem. Then read on to learn about a mathematician that made a similar list. Your list will likely be pretty similar to his.

DISCUSSION

There are many kinds of real-life problems that you may have considered here. One that many of us have dealt with is what kind of car to buy. The first step would be to understand what you need. How many seats do you need? What is your budget? Is gas mileage a priority? What models do you like? This part might be called something like "understand the problem." Did you have a step like this in your list? The second step would be to develop a plan. Where will you look? What research will you do? The third step might be to carry out the plan. Research the best models, look around for the best deals, test drive some models and find the one you like. The next step, hopefully before actually buying the one you have your heart set on, might be to reflect on whether it really meets your needs, fits your budget, and so on. Read on to see how this process is the same for solving a math problem.

POLYA'S FOUR STEPS

George Polya developed a framework that breaks down problem solving into four distinguishable steps. In 1945, he outlined these steps in a now-classic book called *How to Solve It.*

When you approach a problem, if you think that you have to come up with an answer immediately and that there is only one "right" way to reach that answer, a solution may seem to be beyond your grasp. But if you break the problem down and thoughtfully approach each *step* of the problem, it generally becomes more manageable. Polya suggests that you first need to make sure you **understand the problem**. Once you understand the problem, you **devise a plan** for solving the problem. Then you **monitor your plan**; you check frequently to see whether it is productive or is going down a dead-end street. Finally, you **look back at your work**. This last step involves more than just checking your computation; for example, it includes making sure that your answer makes sense. For each of these four steps, there are specific strategies that we will explore in this chapter and that you will refine throughout this course.

Owning versus renting

Instead of just listing Polya's strategies, we are going to discover them by putting them into action. You will notice that I often ask you to stop, think, and write some notes. I really

CLASSROOM CONNECTION

A colleague was working through a word problem with her class one day. Suddenly one of the students said, "But you don't need to do all this stuff you are teaching us; you just know the answer." She was stunned, and discovered that many students believe that the difference between a student and a teacher is that the teacher just knows the answer or automatically knows how to get the answer and thus doesn't need such strategies as guess–check–revise, make a table, draw a diagram, look for patterns, etc. The truth is that we do! Virtually all engineers, scientists, businesspeople, carpenters, researchers, and entrepreneurs approach complex problems by using the very tools that are being stressed in this text.

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mean it! I have come to distinguish between those students who *own* what they learn and those who simply *rent* what they learn. Many students rent what they have learned just long enough to pass the test. However, within days or weeks of the final exam, it's gone, just like a video that has been returned to the store. One of the important differences between owners and renters is that those who own the knowledge tend to be *active readers.*

Using Polya's four steps

I encourage you to use Polya's four steps (on the inside front cover of this book and *Explorations*) in all of the following ways:

1. Use them as a guide when you get stuck.
2. Don't rent them, buy them. Buying them involves paraphrasing my language and adding new strategies that you and your classmates discover. For example, many of my students have added a step to help reduce anxiety: First take a deep breath and remind yourself to slow down!
3. After you have solved a problem, stop and reflect on the tools you used. Over time, you should find that you are using the tools more skillfully.

Think and then read on . . .

Throughout the book, I will often pose a question and ask you to "think and then read on. . . ." Rather than just look to the next paragraph and see the answer, you will learn more if you immediately cover up the next paragraph . . . stop . . . think . . . write down your thoughts . . . and then read on. The phrase "think and read on . . ." is there to remind you to read the book actively rather than passively. An **active reader** stops and thinks about the material just read and asks questions: Does this make sense? Have I had experiences like this? The active reader does the examples with pen or pencil, rather than just reading the author's description.

WHY EMPHASIZE PROBLEM SOLVING?

Although Polya described his problem-solving strategies back in 1945, it was quite some time before they had a significant impact on the way mathematics was taught.

One of the reasons is that until recently, "problems" were generally defined too narrowly. Many of you learned how to do different kinds of problems—mixture problems, distance problems, percent problems, age problems, coin problems—but never realized that they have many principles in common. There has been too great a focus on single-step problems and routine problems. Consider the examples from the National Assessment of Educational Progress shown in Table 1.2.

To solve the first problem, one only has to remember the procedure for finding an average and then use it:

$$\frac{13 + 10 + 8 + 5 + 3 + 3}{6}$$

TABLE 1.2

Problem	Percent correct Grade 11
1. Here are the ages of six children: 13, 10, 8, 5, 3, 3. What is the average age of these children?	72
2. Edith has an average (mean) score of 80 on five tests. What score does she need on the next test to raise her average to 81?	24

Source: Mary M. Lindquist, ed., Results from the *Fourth Mathematics Assessment of the National Assessment of Educational Progress* (Reston, VA: NCTM, 1989), pp. 30, 32.

TABLE 1.3

Textbook word problems	Real-life problems
1. The problem is given.	**1.** Often, you have to figure out what the problem really is.
2. All the information you need to solve the problem is given.	**2.** You have to determine the information needed to solve the problem.
3. There is always enough information to solve the problem.	**3.** Sometimes you will find that there is not enough information to solve the problem.
4. There is no extraneous information.	**4.** Sometimes there is too much information, and you have to decide what information you need and what you don't.
5. The answer is in the back of the book, or the teacher tells you whether your answer is correct.	**5.** You, or your team, decides whether your answer is valid. Your job may depend on how well you can "check" your answer.
6. There is usually a right or best way to solve the problem.	**6.** There are usually many different ways to solve the problem.

OUTSIDE THE CLASSROOM²

Many students still consider the second question to be a "trick" question unless the teacher has explicitly taught them how to solve that kind of problem. However, many employers note that problems that occur in work situations are rarely *just* like the ones in the book. What employers desperately need is more people who can solve the "trick" problems, because, as some may say, "life is a trick problem!"

However, there is no simple formula for solving the second problem. Try to solve it on your own and then read on. . . .

To solve this one, you have to have a better understanding of what an average means. One approach is to see that if her average for 5 tests is 80, then her total score for the 5 tests is 400. If her average for the 6 tests is to be 81, then her total score for the 6 tests must be 486 (that is, 81×6). Because she had a total of 400 points after 5 tests and she needs a total of 486 points after 6 tests, she needs to get an 86 on the sixth test to raise her overall average to 81.

The difference between traditional word problems and many real-life problems

Table 1.3 lists differences between the word problems generally found in textbooks and real-life problems.

When students undertake more authentic problems, they realize that mathematics is more than just memorizing and using formulas, and they begin to value their own thinking.

INVESTIGATION 1.1d Coin Problem

Variations of this problem are often found in elementary school textbooks because it provides an opportunity to move beyond random guess and test.

If 8 coins total 50 cents, what are the coins?

Solve this problem intentionally using and writing out Polya's four steps of problem solving.

Explorations Manual 1.1

DISCUSSION

STEP 1: UNDERSTAND THE PROBLEM

So often students will jump into a problem without stopping to really understand it. Read a problem more than once before attempting to solve. Write down the important information and pay attention to what the question is before starting. Here, you have 8 coins, which might be pennies, nickels, dimes, quarters, or half dollars. All together they equal 50 cents. We need to determine what kind of coins we have.

STEP 2: DEVISE A PLAN

There is more than one strategy to solve any problem. Here we could use a diagram, make a table, use reasoning, or use a bag of coins to help us solve it. Let's consider two strategies of making a diagram and using reasoning.